

Electromagnetic duality in general relativity

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Abstract

By resolving the Riemann curvature relative to a unit timelike vector into electric and magnetic parts, we consider duality relations analogous to the electromagnetic theory. It turns out that the duality symmetry of the Einstein action implies the Einstein vacuum equation without the cosmological term. The vacuum equation is invariant under interchange of active and passive electric parts giving rise to the same vacuum solutions but the gravitational constant changes sign. Further by modifying the equation it is possible to construct interesting dual solutions to vacuum as well as to flat spacetimes.

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1 Introduction

In analogy with the electromagnetic field, it is possible to resolve the gravitational field; i.e. Riemann curvature tensor, into electric and magnetic parts relative to a unit timelike vector [1-4]. In general, a field is produced by its charge (source). Its manifestation when charge is stationary is termed electric, and magnetic when it is moving. The Maxwell electromagnetic field was the first example that brought forth this general feature and hence provided the terminology for other fields. It should be recognised that this is a general property of any classical field.

In general relativity (GR), unlike other fields, charge is also of two kinds. In addition to the usual charge in terms of the usual non-gravitational matter/energy distribution, gravitational field energy itself also has charge. This is what makes the theory non-linear. Thus the electric part would also be of two kinds corresponding to the two kinds of charge, which we term active (non-gravitational energy) and passive (gravitational field energy).

The 20 components of the Riemann curvature are split into 6 each of active (projection of Riemann, $R_{abcd}u^bu^d$) and passive (projection of left and right dual) parts and 8 of magnetic (projection of left or right dual) part. The electric and magnetic parts are second rank 3-space tensors orthogonal to the resolving timelike unit vector u^a , the electric parts are symmetric while the magnetic part is trace free and is the sum of the symmetric Weyl magnetic part and the anti-symmetric part representing energy flux.

Clearly gravitational field has richer structure than electromagnetic field. It would be interesting to see what the duality relation involving electric and magnetic parts implies? It turns out that the duality transformation, analogous to the Maxwell theory, which keeps the Einstein action invariant interestingly implies the Einstein vacuum equation [5-6]. Note that there is a basic difference between gravitational and electromagnetic fields. For the former, the Riemann curvature contains the entire dynamics (field equation) as it involves second order derivatives of the metric (potential), while for the latter fields are first order derivatives of the gauge potential and the dynamics (field equation) would follow from derivatives of the fields. Hence it is understandable that any manipulation of the Riemann curvature would always refer to dynamics of gravitational field.

Remarkably the gravo-electromagnetic duality symmetry of the Einstein action implies the Einstein vacuum equation without the cosmological con-

stant (The equation with cosmological constant is characterized by equality of left and right dual of the Riemann curvature). This property is similar to the other well-known property of GR that the field equation implies the equation of motion for free particles. Now we have the symmetry of the action implying the equation of motion for the field. In GR, there is always synthesis of physical quantities, concepts and equations.

The Einstein vacuum equation, written in terms of electric and magnetic parts, is symmetric in active and passive electric parts. We consider another duality relation, which we call gravo-electric duality, an interchange of active and passive electric parts which keeps the vacuum equation invariant. Under this transformation it turns out that the Ricci and the Einstein tensors are dual to each other. That is, the non-vacuum equation will in general distinguish between active and passive parts. There could occur solutions that are dual to each other [7]. In particular it follows that perfect fluid spacetimes with the equations of state $\rho - 3p = 0$ and $\rho + p = 0$ ($\Lambda \rightarrow -\Lambda$) are self dual while the stiff fluid is dual to dust.

Under the gravo-electric duality though, the vacuum equation remains invariant yielding the same vacuum solutions, but the gravitational constant G will change sign. This is because the scalar curvature R changes sign and now to keep action invariant G must change sign. In obtaining vacuum solutions for isolated bodies, there always remains one equation free which is implied by the others. If we now tamper with this equation the vacuum solution will remain undisturbed but this would make the equation non-invariant under the gravo-electric duality. Thus by modifying the vacuum equation suitably distinct solutions dual to the well-known black hole solutions could be obtained.

In sec. 2, we shall give the electromagnetic decomposition of the Riemann curvature, followed by the duality symmetry of the Einstein action implying the vacuum equation in Sec. 3. In Sec. 4 we shall discuss the duality transformation that keeps the vacuum equation invariant and its implication for the black hole spacetimes. By modifying the vacuum equation it is possible to find interesting solutions dual to the black hole spacetimes which will be discussed in Sec.5. Finally we conclude with discussion.

2 Electromagnetic decomposition

We resolve the Riemann curvature tensor relative to a unit timelike vector as follows :

$$E_{ac} = R_{abcd}u^b u^d, \tilde{E}_{ac} = *R *_{abcd} u^b u^d \quad (1)$$

$$H_{ac} = *R_{abcd}u^b u^d = H_{(ac)} - H_{[ac]}, H_{ca} = R *_{abcd} u^b u^d = \tilde{H}_{ac} \quad (2)$$

where

$$H_{(ac)} = *C_{abcd}u^b u^d \quad (3)$$

$$H_{[ac]} = \frac{1}{2}\eta_{abce}R_d^e u^b u^d. \quad (4)$$

Here C_{abcd} is the Weyl conformal curvature, η_{abcd} is the 4-dimensional volume element. Note that the magnetic part is the projection of left (H_{ac}) or right (H_{ca}) dual and hence either one of them can be taken. We shall therefore drop \tilde{H}_{ac} from further discussion. We have $E_{ab} = E_{ba}, \tilde{E}_{ab} = \tilde{E}_{ba}, (E_{ab}, \tilde{E}_{ab}, H_{ab})u^b = 0$, $H = H_a^a = 0$ and $u^a u_a = 1$. The Ricci tensor could then be written as

$$R_{ab} = E_{ab} + \tilde{E}_{ab} + (E + \tilde{E})u_a u_b - \tilde{E}g_{ab} + \frac{1}{2}H^{mn}u^c(\eta_{acmn}u_b + \eta_{bcmn}u_a) \quad (5)$$

where $E = E_a^a$ and $\tilde{E} = \tilde{E}_a^a$. It may be noted that in view of $G_{ab} = -T_{ab}$, $E = (\tilde{E} + \frac{1}{2}T)/2$ defines the gravitational charge density while $\tilde{E} = -T_{ab}u^a u^b$ defines the energy density relative to the unit timelike vector u^a .

In terms of electromagnetic parts, the vacuum equation $R_{ab} = 0$ would thus read for any unit timelike resolving vector as

$$H_{[ab]} = 0, E \text{ or } \tilde{E} = 0, E_{ab} + \tilde{E}_{ab} = 0. \quad (6)$$

It is symmetric in active and passive electric parts.

3 Gravo-electromagnetic duality

In electromagnetics the duality transformation $\mathbf{E} \rightarrow \mathbf{H}$, $\mathbf{H} \rightarrow -\mathbf{E}$ keeps the Maxwell action, which goes as $E^2 - H^2$, and the source free Maxwell equation invariant. Note that this transformation would essentially lead to vacuous field, and hence there cannot occur a solution obeying it. This is however a symmetry of the action.

In GR, on the other hand, all vacuum solutions would respect the analogous duality transformation because the vacuum equation is implied by the transformation. Analogously we consider

$$E_{ab} \rightarrow H_{ab} \tag{7}$$

$$H_{ab} \rightarrow -\tilde{E}_{ab} \tag{8}$$

which would imply

$$\tilde{E}_{ab} \rightarrow -E_{ab}. \tag{9}$$

The first of the above relations implies that $E = 0$ because $H = 0$ always and $H_{[ab]} = 0$ because E_{ab} and \tilde{E}_{ab} are symmetric. These combined with the third relation are the Einstein vacuum equation (6). The above duality transformation thus implies the vacuum equation and it is a symmetry of the Einstein action as the scalar curvature R remains invariant [5-6]. We thus have a remarkable result:

The above duality transformation is a symmetry of the Einstein action and implies the vacuum equation without the cosmological constant.

The corresponding result in the Maxwell theory is that the duality transformation is only the symmetry of the action and of the source free field equation but it does not imply the field equation. In GR we must recognise the fact that there is a richer structure through two kinds of electric parts and breakup of magnetic part into the symmetric Weyl free-field part and the antisymmetric energy flux part. More importantly these quantities are one order higher in differentiation, as they involve second derivatives of the metric. Hence gravitational electromagnetic parts, unlike the Maxwell case, contain dynamics of the field whereas in the Maxwell case dynamics emerges only on one more differentiation. This is the basic and crucial difference between the two fields. Thus if the duality symmetry of the action were to

imply an equation, it could only be the equation of motion (field equation) for gravitational field.

Note that the cosmological constant cannot appear in the vacuum equation as it is not sustainable by the duality transformation. It could however always come in as matter with its well-known specific equation of state, $\rho + p = 0$. It could however be characterised [1-2] by the following geometric condition,

$$*\mathbf{R} = \mathbf{R}* \quad (10)$$

where $*\mathbf{R}$ and $\mathbf{R}*$ denote respectively the left and right dual of the Riemann curvature. In view of (2), it would imply

$$H_{[ab]} = 0, \quad (11)$$

and because $** = -1$ ($-\mathbf{R} = *\mathbf{R}*$),

$$E_{ab} + \tilde{E}_{ab} = 0. \quad (12)$$

This obviously implies from (5) $R_{ab} = \Lambda g_{ab}$ with $E = \Lambda$. As a matter of fact we can make the following general statement:

The necessary and sufficient condition for $R_{ab} = \Lambda g_{ab}$ is that $\mathbf{R} = \mathbf{R}*$.*

The sufficiency has been shown above. For the necessary condition, substituting eqn.(5) into $R_{ab} = \Lambda g_{ab}$, the eqns. (11) and (12) immediately follow. Further the eqn. (12) means

$$(\mathbf{R} + *\mathbf{R}*) \cdot u \cdot u = 0. \quad (13)$$

Since this is to be true for any arbitrary unit timelike vector, then

$$\mathbf{R} = -*\mathbf{R}* \quad (14)$$

which would imply

$$*\mathbf{R} = \mathbf{R}*. \quad (15)$$

So is proved the necessary condition.

In addition to eqn. (10) if scalar curvature R vanishes, then it is vacuum (because $R = 0$ implies $E = \tilde{E}$ then in view of eqn. (12), $E = \Lambda = \tilde{E} = 0$). The vacuum is thus characterized by $*\mathbf{R} = \mathbf{R}*$ and $R = 0$.

4 Duality transformation and vacuum

For the ready reference, we recall the vacuum equation (6)

$$E \text{ or } \tilde{E} = 0, \quad H_{[ab]} = 0 = E_{ab} + \tilde{E}_{ab} \quad (16)$$

which is symmetric in E_{ab} and \tilde{E}_{ab} .

One may next ask, what keeps the vacuum equation invariant? Clearly the above equation is symmetric between active and passive electric parts. Thus in the second avatar of duality, which is termed as the gravo-electric duality, we define the duality transformation as

$$E_{ab} \longleftrightarrow \tilde{E}_{ab}, \quad H_{ab} \longrightarrow -H_{ab}. \quad (17)$$

Thus the vacuum equation (6) is invariant under the duality transformation (17). The vacuum equation is neutral about the transformation of H_{ab} . As shown below, it turns out that the Weyl electric part changes sign and hence so should the magnetic part. From eqn. (1) it is clear that the duality transformation would map the Ricci tensor into the Einstein tensor and vice-versa. This is because the contraction of Riemann is Ricci while that of its double dual is Einstein. Note also that it maps R to $-R$ because $R = -2(E - \tilde{E})$.

Even though the vacuum equation is a gravo-electric invariant, which would mean the vacuum solutions would also be invariant, the constants of integration may change sign. This is what really happens because the electric part of the Weyl curvature reads as

$$2E_{ab}(W) = E_{ab}(TF) - \tilde{E}_{ab}(TF) \quad (18)$$

where TF stands for trace free part; i.e.

$$E_{ab}(TF) = E_{ab} - \frac{1}{3}Eh_{ab} \quad (19)$$

and

$$h_{ab} = g_{ab} - u_a u_b. \quad (20)$$

Clearly under the duality transformation (17), the Weyl electric part and scalar curvature R change sign. If this were also to be a symmetry of the

Einstein action, the gravitational constant G must also change sign. Of course any symmetry of the equation derived from the action must also be a symmetry of the action itself. That means gravo-electric duality implies $G \rightarrow -G$; i.e. gravity changing its sense! This appears rather strange.

A keener look into what produces active and passive parts does illuminate the situation. E_{ab} is produced by non-gravitational energy distribution while the source for \tilde{E}_{ab} is gravitational field energy (we shall demonstrate this with an example below). The former is always positive as proved by the positive energy theorems while the latter is always negative for an attractive field. Now under duality we interchange active and passive parts which would amount to interchange of the two kinds of energy distributions having inherently opposite signs. This is why G must change sign.

Further the vacuum equation essentially states that the contributions of the two kinds of charge (non-gravitational and gravitational) are on an equal footing and vacuum is characterized by vanishing of their sum. In GR, in contrast to the Newtonian theory, absence of non-gravitational energy distribution alone cannot define vacuum, because of the presence of gravitational field energy which can never be removed. Hence it has to be incorporated with due recognition of its opposite sign. We shall now demonstrate through the well-known case of the Schwarzschild particle that the field energy "curves" space while ordinary matter "pulls" [8].

Consider the spherically symmetric metric,

$$ds^2 = c^2(r, t)dt^2 - a^2(r, t)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (21)$$

It can be easily seen that for this metric $R_{01} = 0, R_0^0 = R_1^1$ lead to $c^2 = a^{-2} = 1 + 2\phi(r)$, and then $R_0^0 = -\nabla^2 \phi$. Thus we again solve the good old Laplace equation rather than contribution of field energy on the right. GR is however supposed to incorporate the contribution of the field energy. What really happens is that the contribution of the field energy is accounted for by the curvature of space leaving the Laplace equation unaltered. This could be readily seen by setting $a = 1$ and then $R_0^0 = 0$ would have the field energy contribution on the right [8]. When $a \neq 1$, $R_0^0 = R_1^1$ washes out the field energy term on the right and gives $ac = 1$. This is how gravitational field energy "curves" space. It is the space curvature which is represented by the passive electric part \tilde{E}_{ab} . The active part is due to space-time curvature which is anchored onto non-gravitational energy distribution. It is well-known that

the Newtonian potential sitting in g_{00} leads to acceleration as gradient of the potential in the geodesic equation.

Thus the interchange of active and passive part under duality would mean interchange of their sources, non-gravitational and gravitational energy, which have opposite sign. Since they have opposite sign, gravity must change its sense and hence $G \rightarrow -G$.

It can be further verified that all the vacuum black hole solutions obey the duality transformation (17) with $G \rightarrow -G$ (for Riemann components see for instance [9]). Of course the charged black hole solution does not obey the transformation implying non-existence of the dual solution. On the other hand de Sitter spacetime is dual to anti de Sitter with $\Lambda \rightarrow -\Lambda$.

The NUT solution could be interpreted as the field of gravito-magnetic monopole [10]. By looking at its electric and magnetic parts [11], one observes that the difference between them goes to zero as $M \rightarrow l, l \rightarrow -M$ where M and l are mass and NUT parameters. Since it is well-known that there cannot exist real self-dual ($\mathbf{R} = *\mathbf{R}$) solutions in GR, the NUT solution could be considered nearest to it as the difference between electric and magnetic parts vanishes for the appropriate transformation of the source parameters. The duality transformation (17) could be cast in the continuous form as follows:

$$E_a^b \longrightarrow E_a^b \cos\theta + \tilde{E}_a^b \sin\theta, \tilde{E}_a^b \longrightarrow E_a^b \sin\theta + \tilde{E}_a^b \cos\theta \quad (22)$$

with

$$\rho \longrightarrow \rho \cos\theta + \frac{1}{2}(\rho + 3p)\sin\theta, p \longrightarrow p \cos\theta + \frac{1}{2}(\rho - p)\sin\theta. \quad (23)$$

The discrete transformation (17) corresponds to $\theta = \pi/2$.

5 Solutions dual to black hole/flat spacetimes

Next the question arises, can we obtain a dual to a vacuum solution? The vacuum equation is symmetric in active and passive parts and hence invariant under the duality transformation (17). However it turns out that in obtaining the well-known black hole solutions not all of the vacuum equations are used. In particular, for the Schwarzschild solution the equation $R_{00} = 0$ in the standard curvature coordinates is implied by the rest of the equations. If we tamper with this equation, the Schwarzschild solution would remain undisturbed for the rest of the set will determine it completely. However this

modification, which does not affect the vacuum solution, breaks the symmetry between active and passive electric parts leading to non-invariance of the modified equation under the duality transformation. This would lead to distinct dual solutions. We shall demonstrate this by obtaining a dual solution to the Schwarzschild solution by modifying the vacuum equation appropriately.

For the metric (21) the natural choice for the resolving vector is of course the hypersurface orthogonal unit vector, pointing along the t -line. From eqn. (6), $H_{[ab]} = 0$ and $E_2^2 + \tilde{E}_2^2 = 0$ lead to $ac = 1$ (for this, no boundary condition of asymptotic flatness need be used [8]). Now $\tilde{E} = 0$ gives $a = (1 - 2M/r)^{-1/2}$, which determines the Schwarzschild solution completely. Note that we did not need to use the remaining equation $E_1^1 + \tilde{E}_1^1 = 0$, it is hence free and is implied by the rest. Without affecting the Schwarzschild solution, we can introduce some distribution in the 1-direction.

We hence write the alternate equation as

$$H_{[ab]} = 0 = \tilde{E}, \quad E_{ab} + \tilde{E}_{ab} = \lambda w_a w_b \quad (24)$$

where λ is a scalar and w_a is a spacelike unit vector along the direction of 4-acceleration. It is clear that it will also admit the Schwarzschild solution as the general solution, and determine $\lambda = 0$. That is, for spherical symmetry the above alternate equation also characterizes vacuum, because the Schwarzschild solution is unique.

Let us now employ the duality transformation (17) to the above equation (22) to write

$$H_{[ab]} = 0 = E, \quad E_{ab} + \tilde{E}_{ab} = \lambda w_a w_b. \quad (25)$$

Its general solution for the metric (21) is given by

$$c = a^{-1} = (1 - 2k - \frac{2M}{r})^{1/2}. \quad (26)$$

This is the Barriola-Vilenkin solution [12] for the Schwarzschild particle with global monopole charge parameter, $\sqrt{2k}$. Again we shall have $ac = 1$ and $E = 0$ will then yield $c = (1 - 2k - 2M/r)^{1/2}$ and $\lambda = 2k/r^2$. This has non-zero stresses given by

$$T_0^0 = T_1^1 = \frac{2k}{r^2}. \quad (27)$$

A global monopole is supposed to be produced by spontaneous breaking of the global symmetry $O(3)$ into $U(1)$ in a phase transition in the early Universe. It is described by a triplet scalar, $\psi^a(r) = \eta f(r)x^a/r$, $x^a x^a = r^2$, which through the usual Lagrangian generates an energy-momentum distribution at large distance from the core precisely of the form given above in (25) [12]. Like the Schwarzschild solution the monopole solution (24) is also the unique solution of eqn.(23).

If we translate eqns. (22) and (23) in terms of the familiar Ricci components, they would read as

$$R_0^0 = R_1^1 = \lambda, R_2^2 = 0 = R_{01} \quad (28)$$

and

$$R_0^0 = R_1^1 = 0 = R_{01}, R_2^2 = \lambda. \quad (29)$$

In either case we shall have $ac = 1$ and $c^2 = f(r) = 1 + 2\phi$, say, and

$$R_0^0 = -\nabla^2 \phi \quad (30)$$

$$R_2^2 = -\frac{2}{r^2}(r\phi)'. \quad (31)$$

Now (26) integrates to give $\phi = -M/r$ and $\lambda = 0$, which is the Schwarzschild solution while (27) will give the dual solution with $\phi = -k - M/r$ and $\lambda = 2k/r^2$, the Schwarzschild solution with global monopole charge. Thus the global monopole owes its existence to the constant k appearing in the solution of the usual Laplace equation. It defines a pure gauge for the Newtonian theory, which could be chosen freely, while the Einstein vacuum equation determines it to be zero. For the dual-vacuum equation (23), it is free like the Newtonian case but it produces non-zero curvature and hence would represent non-trivial physical and dynamical effects (see $R_2^2 = -2k/r^2 \neq 0$ unless $k = 0$). This is the crucial difference between the Newtonian theory and GR in relation to this problem, that the latter determines the relativistic potential ϕ absolutely, vanishing only at infinity. The freedom of choosing

zero of the potential is restored in the dual-vacuum equation, of course at the cost of introducing stresses that represent a global monopole charge. The uniform potential would hence represent a massless global monopole ($M = 0$ in the solution (24)), which is solely supported by the passive part of electric field. As has been argued and shown above, it is the non-linear aspect of the field (which incorporates interaction of gravitational field energy density) that produces space-curvatures and consequently the passive electric part. It is important to note that the relativistic potential ϕ plays the dual role of the Newtonian potential as well as the non-Newtonian role of producing curvature in space. The latter aspect persists even when potential is constant different from zero. It is the dual-vacuum equation that uncovers this aspect of the field.

On the other hand, flat spacetime could also in alternative form be characterized by

$$\tilde{E}_{ab} = 0 = H_{[ab]}, E_{ab} = \lambda w_a w_b \quad (32)$$

leading to $c = a = 1$, and implying $\lambda = 0$. Its dual will be

$$E_{ab} = 0 = H_{[ab]}, \tilde{E}_{ab} = \lambda w_a w_b \quad (33)$$

yielding the general solution,

$$c' = a' = 0 \implies c = 1, a = \text{const.} = (1 - 2k)^{-1/2} \quad (34)$$

which is non-flat and represents a zero mass global monopole, as follows from the solution (24) when $M = 0$. This is also the uniform relativistic potential solution. It can naturally be envisioned as "minimally" curved spacetime, which was first considered by the author [13] long back. Since at that time the stresses given in eqn. (25) did not accord to any acceptable physical distribution, it was not further pursued.

Further it is known that the equation of state $\rho + 3p = 0$, which means $E = 0$, characterizes global texture [14-15]. That is, the necessary condition for spacetimes of topological defects, global textures and monopoles, is $E = 0$. Like the uniform potential spacetime, it can also be shown that the global texture spacetime is dual to flat spacetime. In the above eqns (30) and (31), replace $w_a w_b$ by the projection tensor $h_{ab} = g_{ab} - u_a u_b$. Then non-static homogeneous solution of eqn. (30) is flat while that of the dual-flat

equation (31) is the FRW metric with $\rho + 3p = 0$, which determines the scale factor $S(t) = \alpha t + \beta$, and $\rho = 3(\alpha^2 + k) / (\alpha t + \beta)^2$, $k = \pm 1, 0$. This is also the unique non-static homogeneous solution. The general solutions of the dual-flat equation are thus the massless global monopole (uniform potential) spacetime in the static case and the global texture spacetime in the non-static homogeneous case. Thus they are dual to flat spacetime.

It turns out that spacetimes with $E = 0$ can be generated [16] by considering a hypersurface in 5-dimensional Minkowski space defined, for example, by

$$t^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = k^2(t^2 - x_1^2 - x_2^2 - x_3^2) \quad (35)$$

which consequently leads to the metric

$$ds^2 = k^2 dT^2 - T^2 [d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)]. \quad (36)$$

Here $T^2 = t^2 - x_1^2 - x_2^2 - x_3^2$ and $\rho = 3(1 - k^2)/k^2 T^2$. The above construction will generate spacetimes of global monopole, cosmic strings (and their homogeneous versions as well), and global texture-like type depending upon the dimension and character of the hypersurface. Of course, $E = 0$ always; i.e. zero gravitational mass [16]. The trace of the active part measures the gravitational charge density, responsible for focussing of congruences in the Raychaudhuri equation [17]. The topological defects are thus characterized by vanishing of focussing density (tracelessness of active part).

Application of the duality transformation, apart from vacuum/flat case considered here, has been considered for fluid spacetimes [7]. The duality transformation could similarly be considered for electrovac equations including the Λ -term. Here the analogue of the master equation (23) is

$$H_{[ab]} = 0, E = \Lambda - \frac{Q^2}{2r^4}, E_a^b + \tilde{E}_a^b = (-\frac{Q^2}{r^4} + \Lambda)w_a w^b \quad (37)$$

which has the general solution $c^2 = a^{-2} = (1 - 2k - 2M/r + Q^2/2r^2 - \Lambda r^2/3)$ and $\Lambda = 2k/r^2$. The analogue of eqn. (22) will have $\tilde{E} = -\Lambda - Q^2/2r^4$ instead of E in (35). Thus the duality transformation works in general for a charged particle in the de Sitter universe [18]. Similarly a spacetime dual to the NUT solution has been obtained [19]. In the case of the Kerr solution it turns out, in contrast to others, that the dual solution is not unique. The

dual equation admits two distinct solutions which include the original Kerr solution [20].

6 Discussion

It is remarkable that the Maxwell-like duality transformation of the electric and magnetic parts of gravitational field which is a symmetry of the Einstein action leads to the vacuum field equation. In GR the equation of motion of free particles is implied by the field equation, and now we have the equation of motion of the field being implied by the duality symmetry of the action. Since the Riemann curvature contains the second derivative of the metric, hence the dynamics of the field that would permeate electromagnetic parts as well. Thus a duality relation between electric and magnetic parts that keeps the Einstein action invariant should imply some specific equation between them which could be nothing other than the equation of motion of the field. This property though seems very natural and to some extent obvious has not been, as far as I know, noticed earlier. This is the primary avatar of duality. The another important point to note is that this duality implies the vacuum equation without the cosmological constant. The equation with Λ is characterized by equality of the left and right dual of the Riemann curvature. But it is not a symmetry of the action.

The second avatar is the one that keeps the vacuum equation invariant. It means interchange of active and passive electric parts and we term this gravo-electric duality. Since the equation remains invariant, so would vacuum solutions. It however turns out that the Weyl tensor and scalar curvature change sign. Thus invariance of the action would require G to change sign, implying gravity changing its sense. It has been argued that the sources for active and passive parts are respectively non-gravitational energy and gravitational field energy. It is well-known that they are of opposite sign. Since active and passive parts are interchanged under duality and hence so would be their sources which are of opposite signs. Thus the gravitational constant must change sign and with this all vacuum black hole solutions are self dual.

In the third avatar of duality, we have constructed distinct solutions dual to the well-known black hole solutions by modifying the vacuum equation which

no longer remains invariant under the gravo-electric duality. The modified equation would still admit the unique black hole solutions because the modification is effected in the equation which was free, implied by the other. The dual solutions to Reissner-Nordström, NUT and Kerr black holes have been found [4,18,19,20]. It also turns out that the de Sitter is dual to the anti de Sitter.

Let us consider the simplest and most instructive case of the Schwarzschild solution. As we have seen above, ultimately the vacuum equation reduces to the Laplace equation and its first integral. The latter knocks off the constant of integration in the solution of the former which was free in the Newtonian theory to fix zero of the potential. The Einstein equation does not sustain this freedom and fixes zero at infinity implying asymptotic flatness of the Schwarzschild spacetime. The dual solution on the other hand does nothing else than restoring this constant and breaking asymptotic flatness in the most harmless manner. Of course the spacetime would no longer be empty. The stresses generated by the constant are precisely the same that required for representation of a global monopole charge on a Schwarzschild particle [12]. The dual solutions thus retain the basic physical features of the original vacuum solutions. With the exception of the Kerr solution, the dual solutions to all other black holes are also unique and could be interpreted as black holes with global monopole charge.

In the dual solution if we set the Schwarzschild mass to zero, the resulting spacetime would describe the field of uniform gravitational potential or of zero mass global monopole charge. The spacetime is obviously non-flat. Thus constant relativistic potential has non-trivial dynamics. This is because in GR potential does two things, one as the Newtonian potential as it appears in g_{00} and other the relativistic effect of "curving" space in g_{11} . The former as expected can be transformed away while the latter cannot be even when potential is constant. Thus uniform potential produces non-zero curvature which could be envisioned as an example of "minimum" curvature.

Since duality breaks asymptotic flatness without significantly altering the physical character of the field, it could be the most appropriate way to incorporate Mach's principle. To let the rest of the Universe be non-empty, it is of primary importance to break asymptotic flatness of spacetimes represent-

ing isolated bodies. At the same time the basic character of the field must not change. This is precisely what the dual solution does [18]. Consider the solar system sitting in the uniform potential of our galaxy. The constant in the dual to Schwarzschild solution would then be determined by the uniform galactic potential. In view of this, it can be argued that it is the dual solution that would perhaps describe the solar system more appropriately than the Schwarzschild [8]. The observational support to the Schwarzschild solution will also extend to the dual solution as well [21]. Thus the dual solution would be Machian at the very primary level.

The modified vacuum equation (22) works all right for isolated sources and its dual gives the dual solutions. Can this be a true characterization of vacuum? The equation has $\tilde{E} = 0$ which means vanishing of energy density. This plus absence of energy flux should lead to vacuum. For non-isolated cases, it turns out that eqn.(22) admits the well-known vacuum solutions describing gravitational wave, the Weyl and Levi-Civita metrics. The dual equation (23) too admits the same vacuum solution. They are all self-dual [22].

Finally we would like to say that application of the gravo-electric duality transformation is not confined to GR alone. It can also be applied to construct solutions dual to stringy black holes with dilaton field [23], as well as in the 2+1 gravity [24]. Work on studying its application in other theories is in progress.

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